

On symmetry group of Mollard code*

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Abstract

For a pair of given binary perfect codes C and D of lengths t and m respectively, the Mollard construction outputs a perfect code $M(C, D)$ of length $tm + t + m$, having subcodes C^1 and D^2 , that are obtained from codewords of C and D respectively by adding appropriate number of zeros. In this work we generalize of a result for symmetry groups of Vasil'ev codes [2] and find the group $Stab_{D^2}Sym(M(C, D))$. The result is preceded by and partially based on a discussion of "linearity" of coordinate positions (points) in a nonlinear perfect code (non-projective Steiner triple system respectively).

1 Introduction

There are not so many results on the structure of automorphism group of perfect codes, even in binary case. The investigation of automorphism group and symmetry group of any code is important since these groups are measures of symmetries of the code structure. In the present paper we propose two invariants for measuring the "linearity" of coordinate positions and points in a nonlinear perfect code and non-projective Steiner triple system not necessarily associated with perfect codes. The symmetry group of a perfect code is very closely related to the automorphism group of its Steiner triple system. Beside of the automorphism and symmetry groups, kernel, rank and Steiner triple system of a perfect code, the proposed in the paper these new invariants will be important tools in further research of structural properties of perfect codes.

The well-known result by Phelps [15] states that each finite group is isomorphic to the symmetry group of a perfect binary code, whereas the result of Avgustinovich and Vasil'eva [4] established that the symmetry group of any perfect binary code of length n is isomorphic to the symmetry group of the subcode of all its codewords of weight $(n - 1)/2$. However these results do not give the complete information on the structure of the symmetry and automorphism groups of perfect binary codes. The existence of classes of perfect binary codes with trivial automorphism groups (nonsystematic and systematic) is considered in papers [1, 10, 8]. It is well known [9] that the symmetry group $Sym(H)$ of the Hamming code H of length n is isomorphic to the general linear group $GL(\log(n + 1), 2)$. By the linearity of the Hamming code H of length n , we have

$$|Sym(H)| = |GL(\log(n + 1), 2)| = n(n - 1)(n - 3)(n - 7) \dots (n - (n - 1)/2).$$

The order of the automorphism group of an arbitrary nonlinear perfect binary code was investigated by several authors, see the papers [19, 20, 11, 6, 7]. The main definitions concerning this paper see in [9].

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2 Notations and Definitions

A collection C of binary vectors of length n is called a *perfect* (1-perfect) code if any binary vector is at distance 1 from exactly one codeword of C . The perfect codes have length $n = 2^m - 1$, 2^{n-m} codewords and minimum distance 3. For every admissible n up to equivalence there is the unique linear perfect code of length n , it is called the *Hamming code*. A *Steiner triple system* is a collection of blocks (subsets, called also triples) of size 3 of an n -element set, such that any unordered pair of distinct elements is exactly in one block. The set of codewords of weight 3 in a perfect code C , that contains the all-zero codeword is a Steiner triple system, which we denote $STS(C)$. A Steiner triple system whose linear span is a Hamming code is called *projective*.

With a Steiner triple system S we associate a *Steiner quasigroup* $(P(S), \cdot)$ to be the point set $P(S)$ of S with a binary operation \cdot such that: $i \cdot j = k$, if (i, j, k) is a triple of S and $i \cdot i = i$. A *Steiner loop* $(0 \cup P(S), \star)$ with a binary operation \star fulfills properties $i \star j = k$, if (i, j, k) is a triple of S , $i \star i = 0$ and $i \star 0 = i$.

Let C and D be two binary one-error-correcting codes of lengths t and m respectively. Consider a representation for the Mollard construction [14] for binary codes.

Consider the coordinate positions of the Mollard code $M(C, D)$ of length $tm + t + m$ to be pairs (r, s) from the set $\{0, \dots, t\} \times \{0, \dots, m\} \setminus (0, 0)$.

Let f be an arbitrary function from C to the set of binary vectors of the vector space \mathbf{F}_2^m of length m and $p_1(z)$ and $p_2(z)$ be the generalized parity check functions:

$$p_1(z) = \left(\sum_{s=0}^m z_{1,s}, \dots, \sum_{s=0}^m z_{t,s} \right),$$

$$p_2(z) = \left(\sum_{r=0}^t z_{r,1}, \dots, \sum_{r=0}^t z_{r,m} \right).$$

The binary code $M(C, D) = \{z \in \mathbf{F}_2^{tm+t+m} : p_1(z) \in C, p_2(z) \in f(p_1(z)) + D\}$ is called the Mollard code. In the case when C and D are perfect, the code $M(C, D)$ is perfect. Throughout the paper we consider the case when f is the zero function, C and D are perfect codes, containing the all-zero words $\mathbf{0}^t$ and $\mathbf{0}^m$ respectively.

The Steiner triple system of $M(C, D)$ can be also defined using minimum weight codewords of the initial codes:

$$STS(M(C, D)) = \{x \in \mathbf{F}_2^{tm+t+m} : p_1(x) \in STS(C) \cup \mathbf{0}^t, p_2(x) \in STS(D) \cup \mathbf{0}^m\} \setminus \{\mathbf{0}^{tm+t+m}\}.$$

We use the following convenient partition for the Steiner triple system of Mollard code

$$STS(M(C, D)) = \bigcup_{k,p \in \{0,3\}} T_{kp} \tag{1}$$

where

$$\begin{aligned} T_{00} &= \{((r, 0), (r, s), (0, s)) : r \in \{1, \dots, t\}, s \in \{1, \dots, m\}\}; \\ T_{33} &= \{((r, s), (r', s'), (r'', s'')) : (r, r', r'') \in STS(C), (s, s', s'') \in STS(D)\}; \\ T_{30} &= \{((r, 0), (r', s), (r'', s)) : (r, r', r'') \in STS(C), s \in \{0, \dots, m\}\}; \\ T_{03} &= \{((r, s), (r, s'), (0, s'')) : (s, s', s'') \in STS(D), r \in \{0, \dots, t\}\}. \end{aligned}$$

Let x and y be codewords of C and D respectively. Denote by x^1 and y^2 codewords of $M(C, D)$ such that

$$(x^1)_{r0} = x_r, \text{ for } r \in \{1, \dots, t\} \text{ and } (y^2)_{0s} = y_s, \text{ for } s \in \{1, \dots, m\}$$

with zeros in all positions from $\{0, \dots, t\} \times \{1, \dots, m\}$ and $\{1, \dots, t\} \times \{0, \dots, m\}$ respectively. Note that $M(C, D)$ contains the codes C and D as the subcodes $C^1 = \{x^1 : x \in C\}$ and $D^2 = \{y^2 : y \in D\}$ respectively.

We also use a traditional representation of the Mollard code $M(C, D)$ using its subcodes C^1 , D^2 and $\{e_{r,s} + e_{0,s} + e_{r,0} : r \in \{1, \dots, t\}, s \in \{1, \dots, m\}\}$, where $e_{r,s}$ is a vector of weight one with one in the coordinate position (r, s) :

Lemma 1. *Given a vector $z \in M(C, D)$ there are unique codewords $x \in C$ and $y \in D$ such that*

$$z = x^1 + y^2 + \sum_{(r,s): z_{r,s}=1} (e_{r,s} + e_{0,s} + e_{r,0}).$$

Recall that the dual C^\perp of a code C is a collection of all binary vectors x such that $\sum_{i=1, \dots, n} x_i c_i = 0 \pmod{2}$ for any codeword c of C . For perfect codes C and D , the dual of the Mollard code $M(C, D)$ can be described in the following way:

$$(M(C, D))^\perp = \{z : p_1(z) \in C^\perp, p_2(z) \in D^\perp\}. \quad (2)$$

The rank $\text{rk}(C)$ of a code C is defined to be the dimension of its linear span over \mathbf{F}_2 . The kernel of the code is defined to be the subspace $\text{Ker}(C) = \{x \in C : x + C = C\}$. The rank and kernel are important code invariants. Due to the construction, the Mollard code preserves many properties and characteristics of the initial codes C and D , in particular, we have the iterative formulas for the size of kernel and rank:

$$\dim(\text{Ker}(M(C, D))) = \dim(\text{Ker}(C)) + \dim(\text{Ker}(D)) + tm$$

$$\text{rk}(M(C, D)) = \text{rk}(C) + \text{rk}(D).$$

The symmetry group $\text{Sym}(C)$ of a code C (sometimes being called the permutational automorphism group or full automorphism group [9]) is the subgroup of permutations on n elements preserving the code setwise:

$$\text{Sym}(C) = \{\pi \in S_n : \pi(C) = C\}.$$

The automorphism group of a Steiner triple system of order n is the subgroup of permutations on n elements preserving the collection of blocks of the system.

It is well-known that the symmetry group stabilizes the dual of the code, kernel [16] and its Steiner triple system:

$$\text{Sym}(C) \leq \text{Sym}(\text{Ker}(C)), \quad (3)$$

$$\text{Sym}(C) \leq \text{Sym}(C^\perp), \quad (4)$$

$$\text{Sym}(C) \leq \text{Aut}(\text{STS}(C)). \quad (5)$$

By $\text{Stab}_C G$ and $\text{Stab}_{(C)} G$ of a code C we denote the setwise and codeword-wise stabilizers of the set C by the group G acting on a code C' , $C \subseteq C'$. Let C be a perfect subcode of C' on the nonzero coordinates $N(C)$. We have the following obvious statement.

Proposition 1. *We have that for any perfect code C with a perfect subcode C' on coordinates $N(C)$:*

$$\text{Stab}_{N(C)} \text{Sym}(C') = \text{Stab}_C \text{Sym}(C'), \quad \text{Stab}_{(N(C))} \text{Sym}(C') = \text{Stab}_{(C)} \text{Sym}(C').$$

3 Fundamental partition

Given a perfect code C of length n , one might define *the fundamental partition* associated with C [3], [16] to be the partition of the coordinate set $\{1, \dots, n\}$ into subsets $I_0(C), \dots, I_{2^{n-\text{rk}(C)}-1}(C)$ such that for each $j \in \{0, \dots, 2^{n-\text{rk}(C)}-1\}$ any codeword of the dual code C^\perp has the same values for coordinates with indices from $I_j(C)$ [3]. By $I_0(C)$ we agree to denote the set of coordinates $\{i : x_i = 0 \text{ for all } x \in C^\perp\}$ which is of size $(n+1)/2^{n-\text{rk}(C)}-1$ while $|I_j(C)| = (n+1)/2^{n-\text{rk}(C)}$ for nonzero j . For the proof of the main result we essentially need the following fact:

Lemma 2. [16] *Let $I_0(C), \dots, I_{2^{n-\text{rk}(C)}-1}(C)$ be the fundamental partition associated with a perfect code C , $\pi \in \text{Sym}(C)$. Then*

$$\begin{aligned} \pi(I_0(C)) &= I_0(C), \\ \text{for any } j \in \{1, \dots, 2^{n-\text{rk}(C)}-1\} \text{ there is } j' \text{ such that } \pi(I_j(C)) &= I_{j'}(C). \end{aligned}$$

Avgustinovich et al. [2] considered a fundamental partition to show that any perfect code of rank $n - \log_2(n+1) + 2$ is obtained by Phelps construction. Utilizing a fundamental partition Heden in [6] established an upper bound on the size of the symmetry group of a perfect code as a function of the rank of the code. In Section 5 we apply the idea of the work [6] to prove our result on the symmetry group of a Mollard code.

From the description (2) of $(M(C, D))^\perp$ we obtain the following representation for the fundamental partition associated with the Mollard code $M(C, D)$:

$$\begin{aligned} I_0(M(C, D)) &= (I_0(C) \cup 0) \times (I_0(D) \cup 0) \setminus (0, 0), \\ &= (I_0(C) \cup 0) \times I_{j'}(D), \\ &= I_j(C) \times (I_0(D) \cup 0), \\ &= I_j(C) \times I_{j'}(D), j = 1, \dots, t, j' = 1, \dots, m. \end{aligned}$$

We also use the result of Heden (Lemma 8 of [6]), which provides an inside view on the relationship of the code triples and the elements of the fundamental partition:

Lemma 3. [6] *Let $I_0(C), \dots, I_{2^{n-\text{rk}(C)}-1}(C)$ be the fundamental partition associated with C of length n , and $(\{1, \dots, n\}, \star)$ be the Steiner loop associated with $\text{STS}(M(C, D))$. Then*

1. *for any $j \in \{0, \dots, 2^{n-\text{rk}(C)}-1\}$, $r, r' \in I_j(C)$ we have that $r \star r' \in I_0(C)$;*
2. *for any $j, j' \in \{0, \dots, 2^{n-\text{rk}(C)}-1\}$ there is a unique $j \star' j'$ such that for $r \in I_j(C), r' \in I_{j'}(C)$ we have that $r \star r' \in I_{j \star' j'}(C)$;*
3. *the set $\{I_0(C), \dots, I_{2^{n-\text{rk}(C)}-1}(C)\}$ with respect to the operation \star' is an elementary abelian 2-group.*

4 Linear coordinates

The topic of this section does not concern symmetries of perfect codes directly. Here we discuss the idea of linear coordinates in a perfect code. We consider two characteristics for coordinates of a perfect code or points of a Steiner triple system, which we use later for describing the symmetry groups of Mollard codes or the automorphism groups of Mollard Steiner triple systems. In this section we underline some of their properties and derive an important corollary that we use in the study of the symmetry group of a Mollard code.

The set of the triples of a Steiner triple system

$$\{(i, j, k), (i, a, b), (c, j, a), (c, k, b)\} \tag{6}$$

is called a *Pasch configuration* or, shortly, *Pasch*.

For a Steiner triple system S on elements $\{1, \dots, n\}$ and $i \in \{1, \dots, n\}$, $n \equiv 1, 3 \pmod{6}$, define $\nu_i(S)$ to be the number of different Pasch configurations, incident to i , i. e. such that there are two triples of the Pasch containing the point i .

For a perfect code C of length n and a coordinate position i we consider $\mu_i(C)$ to be the number of code triples from $\text{Ker}(C)$ containing i :

$$\mu_i(C) = |\{x \in \text{STS}(C) \cap \text{Ker}(C) : i \in \text{supp}(x)\}|.$$

Obviously, two coordinate positions i, j of S or C are in different orbits by $\text{Aut}(S)$ or $\text{Sym}(C)$ respectively if $\nu_i(S) \neq \nu_j(S)$ or $\mu_i(C) \neq \mu_j(C)$ respectively. We say that a coordinate i is μ -linear for a code C of length n if $\mu_i(C)$ takes the maximal possible value, i. e. $(n-1)/2$. We say that a point $i \in \{1, \dots, n\}$ is ν -linear for a Steiner triple system S of order n if $\nu_i(S)$ takes the maximal possible value, i. e. $(n-1)(n-3)/4$. By $\text{Lin}_\nu(S)$ and $\text{Lin}_\mu(C)$ denote the sets of ν -linear coordinates of S and μ -linear coordinates of C respectively.

Lemma 4. *Let $\langle \{1, \dots, n\}, \cdot \rangle$ be a quasigroup associated with a Steiner triple system S of any order n . Then the following statements are equivalent:*

1. $l \in \text{Lin}_\nu(S)$;
2. for any distinct $s, s' \in \{1, \dots, n\}$, $s, s' \neq l$ we have $(l \cdot s) \cdot (l \cdot s') = s \cdot s'$;
3. for any distinct $s, s' \in \{1, \dots, n\}$, $s, s' \neq l$ we have $l \cdot (s \cdot s') = (l \cdot s) \cdot s'$.

Proof. A pair of different triples of S , containing $l \in \text{Lin}_\nu(S)$, e.g. $(l, s, l \cdot s)$ and $(l, s', l \cdot s')$ induces the following triples: $(s, s', s \cdot s')$ and $(l \cdot s, l \cdot s', s \cdot s')$. From the last block we have $(l \cdot s) \cdot (l \cdot s') = s \cdot s'$ for any different s and s' if and only if l is ν -linear.

Now consider the triples $(l, s \cdot s', l \cdot (s \cdot s'))$ and $(l, s, l \cdot s)$. The coordinate l is ν -linear for S iff there are triples $(s \cdot s', s, s')$ and $(l \cdot (s \cdot s'), l \cdot s, s')$ in S for any s and s' . Then we see that $l \cdot (s \cdot s') = (l \cdot s) \cdot s'$ iff l is ν -linear. □

The second statement of the previous lemma implies that $0 \cup \text{Lin}_\nu(S)$ is the *nucleus* of a Steiner loop, associated with a Steiner triple system S , which, in particular implies that if S is nonprojective, then $|\text{Lin}_\nu(S)| < (n-1)/2$ (see, for example [22]).

Theorem 1. 1. *Let C be a perfect code. Then we have*

$$\text{Lin}_\mu(C) \subseteq \text{Lin}_\nu(\text{STS}(C)).$$

2. *A subdesign of a Steiner triple system S on $\text{Lin}_\nu(S)$ is a projective Steiner triple system.*
3. *A subcode of a perfect code C on the coordinates $\text{Lin}_\mu(C)$ is a Hamming code.*

Proof. 1. Let i be μ -linear coordinate. Then a pair of triples with the supports $\{i, s, i \cdot s\}$ and $\{i, r, i \cdot r\}$ incident to i can be extended to a Pasch configuration. Indeed, since i is μ -linear, a codeword with the support $\{s, i \cdot s, r, i \cdot r\}$ is in $\text{Ker}(C)$, which, being added with a code triple $\{r, s, r \cdot s\}$ gives a triple $\{i \cdot r, i \cdot s, r \cdot s\}$. Now it is easy to see that the four considered triples define a Pasch.

2. Let l and l' be ν -linear coordinates, $(l, l', l \cdot l')$ be a triple of S . We show that $l \cdot l'$ is ν -linear. Using the equalities from Lemma 4 for any distinct s and s' we have

$$((l \cdot l') \cdot s) \cdot ((l \cdot l') \cdot s') = (l \cdot (l' \cdot s)) \cdot (l \cdot (l' \cdot s')) = (l' \cdot s) \cdot (l' \cdot s') = s \cdot s',$$

which amounts to the fact that $l \cdot l'$ is ν -linear. The subdesign on $\text{Lin}_\nu(S)$ is a projective Steiner triple system, since it is known that a Steiner triple system with the maximum possible number of Pasch configurations is projective [21].

3. Let i and j be two μ -linear coordinates. Then $i \cdot j$ is ν -linear. For any fixed r consider the following Pasch configuration:

$$\begin{array}{ccccc} i \cdot j & (i \cdot j) \cdot r & r & & \\ i \cdot j & j & i & & \\ & r \cdot i & r \cdot i & & \end{array}$$

A triple $(i \cdot j, (i \cdot j) \cdot r, r)$ is in $\text{Ker}(C)$, since three remaining triples in the Pasch-configuration contain i or j . Now from the proven above we see that a subsystem of $STS(C)$ on the points $\text{Lin}_\mu(C)$ is projective, with all its triples being from $\text{Ker}(C)$. Therefore the subcode of $\text{Ker}(C)$ generated by these triples is a Hamming code. \square

Finally, in the case of Mollard codes we have the following formulas for the number of triples in the kernel of Mollard code.

Lemma 5. [13] *Let $M(C, D)$ be a Mollard code obtained from perfect codes C and D of length t and m respectively. Then*

1. $\mu_{(r,0)}(M(C, D)) = \mu_r(C)(m + 1) + m;$
2. $\mu_{(0,s)}(M(C, D)) = \mu_s(D)(t + 1) + t;$
3. $\mu_{(r,s)}(M(C, D)) = 1 + 2(\mu_s(D) + \mu_r(C) + \mu_r(C)\mu_s(D)).$

Corollary 1. *Let $M(C, D)$ be a Mollard code obtained from perfect codes C and D of length t and m respectively. Then $\mu_{(r,s)} = \mu_{(r,0)}$ iff $s \in \text{Lin}_\mu(D) \cup 0$.*

5 The group $\text{Stab}_{D^2}\text{Sym}(STS(M(C, D)))$

For a permutation π on the coordinate positions of the code C (the code D), denote by $\mathcal{D}ub_1(\pi)$ ($\mathcal{D}ub_2(\pi)$ respectively) a permutation of coordinates of $M(C, D)$ such that

$$\mathcal{D}ub_1(\pi)(r, s) = (\pi(r), s) \text{ if } r \text{ is nonzero, } \mathcal{D}ub_1(\pi)(0, s) = (0, s) \text{ otherwise;}$$

$$\mathcal{D}ub_2(\pi)(r, s) = (r, \pi(s)) \text{ if } s \text{ is nonzero, } \mathcal{D}ub_2(\pi)(r, 0) = (r, 0) \text{ otherwise}$$

(see [18], [5]). For a collection Π of permutations we agree that $\mathcal{D}ub_i(\Pi)$ denotes $\{\mathcal{D}ub_i(\pi) : \pi \in \Pi\}$, $i = 1, 2$. We have the following statement:

Lemma 6. *Let C and D be two perfect codes. Then*

$$\begin{aligned} \text{Stab}_{C^1}\text{Sym}(M(C, D)) \cap \text{Stab}_{D^2}\text{Sym}(M(C, D)) = \\ \mathcal{D}ub_1(\text{Sym}(C)) \times \mathcal{D}ub_2(\text{Sym}(D)). \end{aligned}$$

Proof. The inclusion of $\mathcal{D}ub_1(\text{Sym}(C)) \times \mathcal{D}ub_2(\text{Sym}(D))$ into the left hand side of the equality is obvious, see [18]. Let σ be a permutation from $\text{Stab}_{C^1}\text{Sym}(M(C, D)) \cap \text{Stab}_{D^2}\text{Sym}(M(C, D))$. Consider π to be a restriction of σ on the nonzero coordinates of C^1 , i.e. for any $r \in \{1, \dots, t\}$ we have

$$\sigma(r, 0) = (\pi(r), 0),$$

which is equivalent to $\sigma(c^1) = (\pi(c))^1$ for any $c \in C$. We see that $(\pi(C))^1 = \sigma(C^1) = C^1$ amounts to $\pi(C) = C$, so $\pi \in \text{Sym}(C)$. Analogously we have that the restriction of σ on the coordinates of D^2 is a permutation $\pi' \in \text{Sym}(D)$. Note that if $(r, 0)$, $(0, s)$ are fixed by a permutation of coordinates from $M(C, D)$, then the coordinate (r, s) is fixed, since there is a codeword with the support $\{(r, 0), (0, s), (r, s)\}$ in $M(C, D)$. This implies that σ must be equal to $\mathcal{D}ub_1(\pi)\mathcal{D}ub_2(\pi')$. \square

In this section we consider the structure of the setwise stabilizer $Stab_{D^2}Sym(STS(M(C, D)))$ (which we denote in what follows by G) of the subcode D^2 in $Sym(M(C, D))$. The Mollard construction is a generalization of the Vasil'ev construction. In [2] the group of symmetries of Vasil'ev codes is investigated. In this section we obtain an extension of the result for Mollard codes.

Let \mathcal{T} be a subgroup formed by the collection of symmetries τ of G such that

$$\text{for any } r \in \{1, \dots, t\}, \text{ for any } s \in \{0, \dots, m\} \text{ there exists } s' : \tau(r, s) = (r, s'), \quad (7)$$

$$\text{for any } s \in \{1, \dots, m\} \text{ we have } \tau(0, s) = (0, s). \quad (8)$$

From Corollary 1 we obtain:

Lemma 7. *A symmetry $\tau \in \mathcal{T}$ setwise fixes the set of coordinates $\{(r, s) : s \in 0 \cup Lin_\mu(D)\}$ for any $r \in \{1, \dots, t\}$.*

Proposition 2. *The group \mathcal{T} is an elementary abelian 2-group.*

Proof. We show that $\tau \in \mathcal{T}$ is necessarily of order not more than 2. Indeed, let $\tau(r, 0) = (r, s)$, then, taking into account that $\tau(0, s) = (0, s)$, see (8), we have that a triple $\tau((r, 0), (0, s), (r, s)) = ((r, s), (0, s), (r, s'))$ for some s' must be in $STS(M(C, D))$. By (1) the triple $((r, s), (0, s), (r, s'))$ is necessarily in T_{00} , so $s' = 0$, i.e. $\tau(r, s) = (r, 0)$. We see that τ^2 fixes $(r, 0)$ and $(0, s)$ for any $r \in \{1, \dots, t\}, s \in \{1, \dots, m\}$. Therefore, τ^2 must fix (r, s) for any $r \in \{1, \dots, t\}, s \in \{1, \dots, m\}$, because τ^2 fixes elements $(r, 0)$ and $(0, s)$ of the triple $((r, 0), (0, s), (r, s))$. We have shown that τ^2 is an identity. \square

We show that any element of the group G could be represented as a composition of the following three symmetries: $Dub_2(\pi')$, for $\pi' \in Sym(D)$, $Dub_1(\pi)$, for $\pi \in Sym(C)$ and a symmetry $\tau \in \mathcal{T}$. Here $\pi' \in Sym(D)$ is the restriction of σ on the nonzero positions of the subcode D^2 , $\pi \in Sym(C)$ is a permutation, induced by the action of $\sigma Dub_2(\pi'^{-1})$ on the subsets $r \times \{0, \dots, m\}, r = 1, \dots, t$.

Lemma 8. *It is true that*

1. $Stab_{(C^1)}G = Dub_2(Sym(D)) \triangleleft G$;
2. $Stab_{(D^2)}G = \{Dub_1(\pi)\tau : \pi \in Sym(C), \tau \in \mathcal{T}\} \triangleleft G$;
3. $G = Dub_2(Sym(D)) \times \{Dub_1(\pi)\tau : \pi \in Sym(C), \tau \in \mathcal{T}\}$.

Proof. Let σ be from G . We have that $\sigma(D^2) = D^2$, so the restriction of σ on D^2 is a permutation $\pi' \in Sym(D)$ (see the proof of Lemma 6).

We now show that $\sigma' = \sigma Dub_2(\pi'^{-1})$ acts on the following subsets of coordinates: $r \times \{0, \dots, m\}, r \in \{1, \dots, t\}$. For any $s \in \{1, \dots, m\}$ let $\sigma'(r, 0)$ be (r', s') and $\sigma'(r, s)$ be (r'', s'') for some s', s'' and nonzero r', r'' . Since $((r, 0), (r, s), (0, s))$ is a triple of $M(C, D)$, so must be $(\sigma'(r, 0), \sigma'(r, s), \sigma'(0, s)) = ((r', s'), (r'', s''), (0, s))$. From (1) the triple $((r', s'), (r'', s''), (0, s))$ is in T_{00} or T_{03} and both cases necessarily imply that $r' = r''$.

So, there is a permutation π of coordinate positions of the code C such that

$$\sigma'(r \times \{0, \dots, m\}) = \pi(r) \times \{0, \dots, m\}, \quad (9)$$

for $r \in \{1, \dots, t\}$. The permutation π is necessarily from $Sym(C)$ since σ' should act as an element of $Sym(C)$ on the first coordinates of the subcode C^1 . For any $x \in C$ we have that

$$p_1(\sigma'(x^1)) = p_1(\pi(x)^1) = \pi(x) \in C,$$

which is true iff $\pi \in Sym(C)$.

Therefore σ is $Dub_2(\pi')Dub_1(\pi)\tau$ for some $\tau \in \mathcal{T}$. By definition of \mathcal{T} , see (8), the groups \mathcal{T} and $Dub_1(Sym(C))$ are subgroups of $Stab_{(D^2)}G$, $Dub_2(Sym(D))$ is a subgroup of $Stab_{(C^1)}G$. Since a pointwise stabilizer of a group G acting on a set is a normal subgroup of G , we obtain the required. \square

By Lemma 8 we are now focused on the description of \mathcal{T} . In the next lemma we use the idea similar to that of work [6].

Lemma 9. *Let $(\{I_j(C), j = 0, \dots, 2^{t-\text{rk}(C)} - 1\}, \star')$ be a Steiner loop associated with the elements of the fundamental partition of the code C , $(0 \cup \text{Lin}_\mu(D), \star)$ be a subloop of Steiner loop associated with $\text{STS}(D)$ and τ be a symmetry of \mathcal{T} . Then there is a group homomorphism $\alpha : (\{I_j(C), j = 0, \dots, 2^{t-\text{rk}(C)} - 1\}, \star') \rightarrow (0 \cup \text{Lin}_\mu(D), \star)$, such that*

$$\tau(r, s) = (r, s \star \alpha(j)),$$

where $r \in I_j(C)$.

Proof. For any $r \in \{1, \dots, t\}$ define l_r from the condition $\tau(r, 0) = (r, l_r)$. By Lemma 7, the coordinate l_r must be in $0 \cup \text{Lin}_\mu(D)$. Consider a triple $((r, s), (0, s), (r, 0))$. Since $\tau((r, s), (0, s), (r, 0))$ is a triple of $M(C, D)$ and $\tau(r, 0) = (r, l_r)$, $\tau(r, s) = (r, s')$, $\tau(0, s) = (0, s)$, we see that $p_2(e_{r, l_r} + e_{r, s'} + e_{0, s}) = e_{l_r} + e_{s'} + e_s$ must be in D , so either one of the elements l_r, s' is zero and the remaining is equal to s or $(l_r, s, s') \in \text{STS}(D)$. This could be rewritten in the form $s' = s \star l_r$ and we have

$$\tau(r, s) = (r, s \star l_r). \quad (10)$$

If r is zero we set l_0 equal to 0 according to (8).

We prove that $l_r = l_{r'}$, for $r, r' \in I_j(C)$. Consider the restriction τ' of τ on the perfect subcode $M(C, D_\mu)$ of the code $M(C, D)$, where D_μ is a linear subcode of D on the positions $\text{Lin}_\mu(D)$. The restriction is correct, i. e. τ' is a symmetry of $M(C, D_\mu)$, since by Lemma 7 a symmetry τ fixes the set of the coordinates of $M(C, D_\mu)$. We have the following representation for the fundamental partition associated with $M(C, D_\mu)$ (see Section 3):

$$\begin{aligned} I_0(M(C, D_\mu)) &= I_0(C) \times 0, \\ I_j(C) \times s, &\text{ for all } s \in \{0, \dots, m\}, \\ (I_0(C) \cup 0) \times s &\in \{1, \dots, m\}. \end{aligned}$$

By (7) and Lemma 2 we see that τ' fixes any element $(r, 0)$ of $I_0(M(C, D_\mu))$, so we have that l_r is equal to 0 for all $r \in I_0(C)$.

For any distinct r, r' , we have $r \cdot r' = r \star r'$ and

$$\tau((r, 0), (r', 0), (r \star r', 0)) = ((r, l_r), (r', l_{r'}), (r \star r', l_{r \star r'})). \quad (11)$$

If $r, r' \in I_j(C)$, $j \in \{0, \dots, 2^{t-\text{rk}(C)} - 1\}$ then $r \star r' = r \cdot r'$ is in $I_0(C)$ (see Lemma 3) so $l_{r \star r'} = 0$ and (11) implies that $l_r = l_{r'}$. Therefore the action of τ can be presented as $(r, s) \rightarrow (r, s \star \alpha(j))$ if $r \in I_j(C)$ for some mapping α of $\{I_j(C), j = 0, \dots, 2^{t-\text{rk}(C)} - 1\}$ into $0 \cup \text{Lin}_\mu(D)$.

Moreover, we have that α is an operation-preserving mapping. By Lemma 3 for any j, j' there is a unique $j \star j'$ such that for $r \in I_j(C)$, $r' \in I_{j'}(C)$, $r \star r'$ is in $I_{j \star j'}(C)$. Because any triple $(\tau(r, 0), \tau(r', 0), \tau(r \star r', 0)) = ((r, l_r), (r', l_{r'}), (r \star r', l_{r \star r'}))$ must be a triple of $\text{STS}(M(C, D))$ we necessarily have that $\alpha(j) \star \alpha(j') = l_r \star l_{r'} = l_{r \star r'} = \alpha(j \star j')$. \square

Now from Lemma 9 we immediately obtain an evaluation for the order of \mathcal{T} .

Corollary 2. *The order of \mathcal{T} is not more than $(1 + |\text{Lin}_\mu(D)|)^{t-\text{rk}(C)}$.*

For a codeword $u \in C$ and an element $l \in \text{Lin}_\mu(D)$, denote by $\text{Ort}_l(u)$ the permutation on the coordinates of $M(C, D)$ defined in the following way

$$\begin{aligned}\text{Ort}_l(u)(r, s) &= (r, s \star l), \text{ for } r \in \text{supp}(u), s \in \{0, \dots, m\}, \\ \text{Ort}_l(u)(r, s) &= (r, s), \text{ otherwise,}\end{aligned}$$

where \star is a binary operation in the Steiner loop associated with $\text{STS}(D)$.

We agree that $\text{Ort}_A(U)$ denotes the collection of permutations $\{\text{Ort}_l(u) : l \in A, u \in U\}$.

Lemma 10. *Let C and D be perfect codes. Then $\langle \text{Ort}_{\text{Lin}_\mu(D)}(C^\perp) \rangle \leq \mathcal{T}$ and $\langle \text{Ort}_{\text{Lin}_\mu(D)}(C^\perp) \rangle \cong Z_2^{(\log_2(1+|\text{Lin}_\mu(C)|))^{t-\text{rk}(C)}}$, here t is length of the code C .*

Proof. Let u be an arbitrary nonzero vector from C^\perp , $l \in \text{Lin}_\mu(D)$, z be an arbitrary codeword of $M(C, D)$. We show that $\text{Ort}_l(u)$ is in $\text{Sym}(M(C, D))$.

By definition of $\text{Ort}_l(u)$, $p_1(\text{Ort}_l(u)(z)) = p_1(z)$. Using Lemma 1,

$$z = x^1 + y^2 + \sum_{(r,s): z_{r,s}=1} (e_{r,s} + e_{0,s} + e_{r,0})$$

for some $x \in C$ and $y \in D$. Therefore we have the following equality:

$$p_2(\text{Ort}_l(u)(z)) = p_2(\text{Ort}_l(u)(x^1)) + p_2(\text{Ort}_l(u)(y^2)) + p_2\left(\sum_{(r,s): z_{r,s}=1} (e_{r,s} + e_{0,s} + e_{r,0})\right). \quad (12)$$

We show that the righthanded side of (12) is a codeword of D . By definition of $\text{Ort}_l(u)$, we have that $\text{Ort}_l(u)(y^2) = y^2$ and therefore $p_2(\text{Ort}_l(u)(y^2)) = y$. Since $u \in C^\perp$, there is the vector with the support $\text{supp}(u) \times \{0, \dots, m\}$ in $(M(C, D))^\perp$, see (2). Then the size of $\text{supp}(x^1) \cap (\text{supp}(u) \times \{0, \dots, m\})$ must be even. Since $\text{supp}(x^1) = \text{supp}(x) \times \mathbf{0}^{tm}$, we have that

$$\text{supp}(x^1) \cap (\text{supp}(u) \times \{0, \dots, m\}) = (\text{supp}(x) \cap \text{supp}(u)) \times \mathbf{0}^{tm}.$$

Since $\text{Ort}_l(u)(x^1)$ is obtained from x^1 by interchanging the subset of zero coordinates $\text{supp}(u) \times i$ and the coordinates from the subset $\text{supp}(u) \times \mathbf{0}^{tm}$, we see that $p_2(\text{Ort}_l(u)(x^1))$ is zero, since the block $\text{supp}(u) \times i$ contains even number of ones in $\text{Ort}_l(u)(x^1)$. So, $p_2(\text{Ort}_l(u)(x^1))$ is zero.

Now, by definition of $\text{Ort}_l(u)$ the triple $e_{r,s} + e_{0,s} + e_{r,0}$ is fixed by $\text{Ort}_l(u)$ for any $r \notin \text{supp}(u)$ and any $s \in \{1, \dots, m\}$ and therefore $p_2(\sigma(e_{r,s} + e_{0,s} + e_{r,0})) = \mathbf{0}^m$. If r is in $\text{supp}(u)$, then $\sigma(e_{r,s} + e_{0,s} + e_{r,0}) = e_{r,s \star l} + e_{0,s} + e_{r,l}$ and so we have that

$$p_2(\sigma(e_{r,s} + e_{0,s} + e_{r,0})) = e_{s \star l} + e_s + e_l \in D$$

by definition of the operation \star . Combining the obtained values for the righthand side of the equality (12) we have

$$p_2(\text{Ort}_l(u)(z)) = y + \sum_{z_{r,s}=1, r \in \text{supp}(u)} (e_{s \star l} + e_s + e_l).$$

Any triple in the last sum is from $\text{Ker}(D)$, since it contains $l \in \text{Lin}_\mu(D)$, so we obtain that $p_2(\text{Ort}_l(u)(z))$ is in D . Therefore $\text{Ort}_l(u)$ is a symmetry of $M(C, D)$.

By Proposition 2, we see that $\langle \text{Ort}_{\text{Lin}_\mu(D)}(C^\perp) \rangle$ is an elementary abelian 2-group. A minimum set of generators for this group could be chosen to consist of symmetries $\text{Ort}_l(c)$, where l runs through a minimal generator set for the elementary abelian 2-group associated to the projective Steiner triple subsystem of $\text{STS}(D)$, defined on the points $\text{Lin}_\mu(D)$, and c runs through a set of generators of the code C^\perp . Therefore we have that

$$\langle \text{Ort}_{\text{Lin}_\mu(D)}(C^\perp) \rangle \cong Z_2^{(\log_2(1+|\text{Lin}_\mu(C)|))^{t-\text{rk}(C)}}.$$

□

By Corollary 2 and Lemma 10 we have the description for G :

Theorem 2. *Let C and D be two reduced perfect codes. Then*

$$G = (\mathcal{D}ub_1(\text{Sym}(C)) \ltimes \langle \text{Ort}_{\text{Lin}_\mu(D)}(C^\perp) \rangle) \times \mathcal{D}ub_2(\text{Sym}(D)).$$

With a slightly shorter proof than that for the previous theorem we obtain the analogous result for Steiner triple systems:

Theorem 3. *Let S_1 and S_2 be arbitrary two Steiner triple systems, $M(S_1, S_2)$ be a Steiner triple system obtained from S_1 and S_2 by applying the Mollard construction. Then*
 $\text{Stab}_{S_2^2} \text{Aut}(M(S_1, S_2)) = (\text{Dub}_1(\text{Aut}(S_1)) \ltimes \langle \text{Ort}_{\text{Lin}_\nu(S_2)}(S_1^\perp) \rangle) \times \text{Dub}_2(\text{Aut}(S_2)).$

In work [13] a class of Mollard codes with symmetry groups, fixing D^2 fulfilling special algebraic properties was obtained. By Theorem 2 we have a description for the symmetry groups of this class.

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